Analysis and Control of Linear Time-Varying (LTV) Systems

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Abstract

Consider a linear time-varying (LTV) system described by the state-space equation $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}(t)$. The main objectives of this paper include (**i**) determination of the analytical or closed-form solutions for the fundamental matrix $\mathbf{X}(t)$ and the state transition matrix $\mathbf{\Phi}(t, t_o)$ of the LTV system; (**ii**) design of feedback control, such that the closed-loop system matrix $\mathbf{A}_{cl}(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)$, where $\mathbf{K}(t)$ is a gain matrix, has desirable properties, in particular, $\mathbf{A}_{cl}(t)$ being commutative and triangular; and (**iii**) design of observers such that the observer matrix $\mathbf{A}_o(t)=\mathbf{A}(t)-\mathbf{H}(t)\mathbf{C}(t)$, where $\mathbf{H}(t)$ is the observer gain matrix, has desirable properties as in (**ii**), namely, $\mathbf{A}_o(t)$ being commutative and triangular. The commutativity and triangularization of $\mathbf{A}_{cl}(t)$ and $\mathbf{A}_o(t)$ facilitate the analytical solutions for their fundamental and state transition matrices. Examples and simulations demonstrate the design objectives.

Key Words: Linear time-varying (LTV), feedback, controller, observer, commutativity, triangular, separation principle, triangulatiaon, commutativity.

1 Linear-Time-Varying (LTV) Systems

Consider an *n*th-order linear time-varying (LTV) system described by the state-space equation:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_o) = \mathbf{x}_o, \tag{1a}$$

$$\mathbf{y} = \mathbf{C}(t)\mathbf{x}, \tag{1b}$$

where $\mathbf{x}(t)$ is an $n\times 1$ state vector, $\boldsymbol{u}(t)$ an $\ell\times 1$ control vector, and $\mathbf{y}(t)$ an $m\times 1$ output vector; $\mathbf{A}(t)$, $\mathbf{B}(t)$ and $\mathbf{C}(t)$ are, respectively, $n\times n$, $n\times\ell$ and $m\times n$ time-varying matrices, and $\mathbf{x}(t_o) = \mathbf{x}_o$ is the initial condition. Using the method of variation of parameters, the solution of (1) can be expressed as

$$\mathbf{x}(t) = \mathbf{\Phi}_A(t, t_o)\mathbf{x}(t_o) + \int_{t_o}^{t} \mathbf{\Phi}_A(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau, \tag{2a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \tag{2b}$$

where $\mathbf{\Phi}_A(t, t_o)$ denotes the $n\times\ n$ state transition matrix associated with $\mathbf{A}(t)$, and satisfies

$$\dot{\mathbf{\Phi}}_A(t, t_o) = \mathbf{A}(t)\,\mathbf{\Phi}_A(t, t_o), \quad \mathbf{\Phi}_A(t_o, t_o) = \mathbf{I}_n$$

$\mathbf{I}_n$ denotes the $n\times n$ unit matrix. (3)

The state transition matrix $\mathbf{\Phi}_A(t,\tau)$ is related to the fundamental matrix $\mathbf{X}_A(t)$ by

$$\mathbf{\Phi}_A(t,\tau) = \mathbf{X}_A(t)\mathbf{X}_A^{-1}(\tau), \tag{4}$$

where the fundamental matrix

$$\mathbf{X}_A(t) = \mathrm{e}^{\int_{t_o}^{t}\mathbf{A}(\tau)d\tau} \tag{5}$$

solves the $n\times\ n$ matrix differential equation

$$\dot{\mathbf{X}}_A(t) = \mathbf{A}(t)\,\mathbf{X}_A(t), \quad \mathbf{X}_A(t_0) = \mathbf{X}_o. \tag{6}$$

Further, the matrix exponential in (5) is defined by the power series $\sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_{t_o}^{t}\mathbf{A}(\tau)d\tau\right)^k$, thereby (5) yields, in general, a solution of the form

$$\mathbf{X}_A(t) = \exp\left(\int_{t_o}^{t}\mathbf{A}(\tau)d\tau\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_{t_o}^{t}\mathbf{A}(\tau)d\tau\right)^k. \tag{7}$$

However, if $\left(\int_{t_o}^{t}\mathbf{A}(\tau)d\tau\right)^{m+1} = \mathbf{0}$ for some finite $m < \infty$, then (7) becomes a finite-sum solution given by [18]

$$\mathbf{X}_A(t) = \mathbf{I}_n + \int_{t_o}^{t}\mathbf{A}(\tau)d\tau + \cdots + \frac{1}{m!}\left(\int_{t_o}^{t}\mathbf{A}(\tau)d\tau\right)^m. \tag{8}$$

It is well known that determining the matrix exponential given by (5) is a difficult task, even for constant matrix $\mathbf{A}$ [13]. Note also that $\mathbf{X}_A(t)$ is nonsingular for all *t*, but may be nonunique; however, the state transition matrix given by $\mathbf{\Phi}_A(t, t_o) = \mathbf{X}_A(t)\mathbf{X}_A^{-1}(t_o)$ is unique for all t_o and *t*.

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2 Facts about LTV Systems

Consider an LTV system described by the homogenous ordinary differential equation (ODE)

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}\,, \quad \mathbf{x}(t_o) = \mathbf{x}_o\,. \tag{9}$$

The following are important facts about the fundamental matrix $\mathbf{X}_A(t)$ and state transition matrix $\mathbf{\Phi}_A(t, t_o)$ associated with $\mathbf{A}(t)$:

F1: Sufficient Conditions for the Existence of Fundamental and State Transition Matrices

The conditions are summarized in Table 1 ([2, 7, 9, 12, 14-16]).

Table 1: Sufficient conditions for the existence of analytical solutions of $\mathbf{X}_A(t)$ and $\mathbf{\Phi}_A(t, t_o)$

(i)	$\mathbf{A}(\tau)$ has piecewise continuous elements $\{a_{ij}(\tau)\}$ for all $i,\ j$ and $\tau \in [t_o,\ t]$;
(ii)	$\mathbf{A}(t)$ commutes with its integral $\int_{t_o}^{t} \mathbf{A}(\tau)d\tau$, i.e., $\mathbf{A}(t)\left(\int_{t_o}^{t} \mathbf{A}(\tau)d\tau\right) = \left(\int_{t_o}^{t} \mathbf{A}(\tau)d\tau\right)\mathbf{A}(t)$;
(iii)	$\mathbf{A}(t)\mathbf{A}(s)=\mathbf{A}(s)\mathbf{A}(t)$ for all t and s;
(iv)	$\int_{t_o}^{t_1} \mathbf{A}(\tau)d\tau \int_{t_o}^{t_2} \mathbf{A}(s)ds = \int_{t_o}^{t_2} \mathbf{A}(s)ds \int_{t_o}^{t_1} \mathbf{A}(\tau)d\tau$;
(v)	$\mathbf{A}(t) = \alpha(t)\mathbf{A}$, where $\alpha(t)$ is a scalar function and $\mathbf{A}$ is a constant matrix;
(vi)	$\mathbf{A}(t) = \sum_{i=1}^{m} \alpha_i(t)\mathbf{A}_i$, where $\alpha_i(t)$ are scalar functions, and $\mathbf{A}_i$ are constant matrices such that $\mathbf{A}_i\mathbf{A}_j=\mathbf{A}_j\mathbf{A}_i$, i.e., $\mathbf{A}_i$ and $\mathbf{A}_j$ commute for all $\{i, j\}=\{1, 2, \cdots, m\}$;
(vii)	$\mathbf{A}(t)$ can be diagonalized as $\mathbf{D}(t) = \mathbf{T}(t)^{-1}\mathbf{A}(t)\mathbf{T}(t)$, where $\mathbf{D}(t) = diag(\lambda_1(t), \ldots, \lambda_n(t))$ and $\{\lambda_1(t), \ldots, \lambda_n(t)\}$ denote the eigenvalues of $\mathbf{A}(t)$. $\mathbf{T}(t)$ is the similarity transformation matrix.

F2: Properties of Diagonal, Upper and Lower Triangular Matrices

A general upper triangular matrix $\mathbf{U}(t)$ and a lower triangular matrix $\mathbf{L}(t)$ have the forms, respectively,

$$\mathbf{U}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ 0 & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}(t) \end{bmatrix} \quad \text{and}$$

$$\mathbf{L}(t) = \begin{bmatrix} a_{11}(t) & 0 & \cdots & 0 \\ a_{21}(t) & a_{22}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{2n}(t) & \cdots & a_{nn}(t) \end{bmatrix} \tag{10}$$

where $\{a_{ij}\}$ are the elements. Their properties help find the corresponding fundamental and state transition matrices $\mathbf{X}(t)$ and $\mathbf{\Phi}(t, 0)$:

Properties of upper diagonal matrix $\mathbf{U}$ and lower diagonal matrix $\mathbf{L}$:

- $\mathbf{U}^T = \mathbf{U}$ and $\mathbf{L}^T = \mathbf{L}$, where $(.)^{\mathrm{T}}$ denotes the transpose of (.);
- The product of two upper triangular or lower triangular matrices is, respectively, an upper triangular or lower triangular matrix, i.e., $\mathbf{U}_1\mathbf{U}_2 = \mathbf{U}$ and $\mathbf{L}_1\mathbf{L}_2 = \mathbf{L}$;
- A diagonal matrix $\mathbf{D}(t)$ is invertible if and only if all its diagonal elements are nonzero;
- Diagonal matrices $\mathbf{D}(t)$ always commute, i.e., $\mathbf{D}(t)\mathbf{D}(s)=\mathbf{D}(s)\mathbf{D}(t)$;
- Upper and lower triangular matrices with identical diagonal elements are commutative; furthermore, if all their diagonal elements are zeros, they become nilpotent matrices. In addition, the eigenvalues of upper and lower triangular matrices are equal to their diagonal elements.
- The eigenvalues of an $n \times n$ time-varying matrix $\mathbf{A}(t)$ can be determined by using the conventional characteristic equation, i.e., $\Delta = \det[\lambda(t)\mathbf{I}_n - \mathbf{A}(t)] = 0$ [15].

F3: Method of Superposition Principle (MSP)

An $n \times n$ analytical solution $\mathbf{X}(t)$ for the homogenous LTV system described by ODE (6) can be constructed by picking n linearly independent initial conditions [1-3]. The method is summarized as follows:

Set n normalized independent initial condition (IC) as:

$$\mathbf{x}^1(t_o) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \mathbf{x}^2(t_o) = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \ \mathbf{x}^n(t_o) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{11}$$

Let $\{\mathbf{x}^i(t),\ i = 1,\ \cdots,\ n\}$ be the solution of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ based on (10), i.e., $\mathbf{x}^i(t)$ is solved uniquely one column at a time for each initial condition $\mathbf{x}^i(t_o)$. This construction is based on the principle of superposition of linear systems, and is a *method of superposition principle (MSP)* with normalized ICs. The resulting $n\times\ n$ nonsingular matrix, denoted by $\mathbf{X}_{Normalized}(t)$, is given by

$$\mathbf{X}_{Normalized}(t) = \left[\mathbf{x}^1(t) \;\vdots\; \cdots \;\vdots\; \mathbf{x}^n(t)\right]. \tag{12}$$

It follows from (12) that the normalized fundamental matrix $\mathbf{X}_{Normalized}(t)$ satisfies

$$\dot{\mathbf{X}}_{Normalized}(t) = \left[\dot{\mathbf{x}}^1(t) \;\vdots\; \cdots \;\vdots\; \dot{\mathbf{x}}^n(t)\right] = \mathbf{A}(t)\left[\mathbf{x}^1(t) \;\vdots\; \cdots \;\vdots\; \mathbf{x}^n(t)\right] = \mathbf{A}(t)\mathbf{X}_{Normalized}(t). \tag{13}$$

The simple initial condition (10) is a convenient choice to determine $\mathbf{X}_{Normalized}(t)$. Note that it is also possible to choose any set of initial conditions $\left\{\overline{\mathbf{x}}^i(t_o),\ i=1,\ 2,\ \cdots,\ n\right\}$, thereby obtaining a different $\overline{\mathbf{X}}(t)$, as long as $\left\{\overline{\mathbf{x}}^i(t_o),\ i=1,\ 2,\ \cdots,\ n\right\}$ are linearly independent vectors. Therefore, the non-uniqueness of $\mathbf{X}_{Normalized}(t)$ and $\overline{\mathbf{X}}(t)$ provides flexibility for analyzing LTV systems; however, the state transition matrix $\mathbf{\Phi}(t,t_o)$ is unique and is given by (4).

Remark 1: The method of superposition principle (MSP) hinges on $\mathbf{A}(t)$ being an upper or lower triangular matrix [6], so that the analytical solution of each ODE $\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)x_j$ for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$ can be determined successively. The method is particularly attractive for **manual calculations** of $\mathbf{X}(t)$ and $\mathbf{\Phi}(t,t_o)$ for low dimensional systems, such as $n = 2$ and $n = 3$.

F4: Matrix Exponentials and Commutativity of Constant and Time-Varying Matrices

Given two square matrices **A** and **B**, it follows, in general, that [5, 7]

$$e^{\mathbf{A}t}\mathbf{B} \neq \mathbf{B}e^{\mathbf{A}t}, \tag{14a}$$

$$e^{\mathbf{A}t}e^{\mathbf{B}t} \neq e^{\mathbf{B}t}e^{\mathbf{A}t}, \tag{14b}$$

$$e^{\mathbf{A}t}e^{\mathbf{B}t} \neq e^{(\mathbf{A}+\mathbf{B})\mathbf{t}}. \tag{14c}$$

Equality will hold if and only if **A** and **B** commute, i.e., **AB**=**BA**. Further, if **AB**=**BA**, then $e^{\mathbf{A}t}e^{\mathbf{B}t} = e^{\mathbf{B}t}e^{\mathbf{A}t} = e^{\mathbf{A}t+\mathbf{B}t} \Rightarrow e^{\mathbf{A}t} = e^{\mathbf{A}t+\mathbf{B}t-\mathbf{B}t}$. In addition, if **Y** is invertible, then $e^{\mathbf{YXY}^{-1}} = \mathbf{Y}e^{\mathbf{X}}\mathbf{Y}^{-1}$ [11].

F5: Separation Principle for LTI and LTV Systems

2.1 LTI Systems

Consider the LTI system

$$\dot{\mathbf{x}} = \mathbf{Ax+Bu},\ \mathbf{x(0)=xo}, \tag{15a}$$

$$\mathbf{y = Cx}, \tag{15b}$$

where **A**, **B** and **C** are constant matrices of compatible dimensions. It follows that the pair [**A**, **B**] is controllable if and only if the pair [$\mathbf{A}^T$, $\mathbf{B}^T$] is observable, which is the well-known property of duality for control and estimation.

The feedback control **u** is assumed to be given by

$$\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}}, \tag{16}$$

where **K** is the feedback gain matrix and $\hat{\mathbf{x}}$ is an estimate of **x** generated by an observer.

Next, an observer for (17) can be constructed as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{Bu} + \mathbf{H}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) = (\mathbf{A} - \mathbf{BK} - \mathbf{HC})\hat{\mathbf{x}} + \mathbf{Hy}, \tag{17}$$

where $\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}}$ given by (18), and **H** is the observer gain matrix.

Define the estimation error as

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} \quad (\Rightarrow \quad \mathbf{x} = \hat{\mathbf{x}} + \mathbf{e}), \tag{18}$$

which yields

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}. \tag{19}$$

Further, (19) can be expressed as ■

$$\dot{\mathbf{e}} = (\mathbf{Ax} + \mathbf{Bu}) - \left[(\mathbf{A} - \mathbf{HC})\hat{\mathbf{x}} + \mathbf{Bu} + \mathbf{HCx}\right] = (\mathbf{A\text{-}HC})\mathbf{e}. \tag{20}$$

Also using (18), (20) can be expressed as

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BKe}. \tag{21}$$

Combing (20) and (21), we obtain the augmented systems

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{e}}\end{bmatrix} = \begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{BK}\\ \mathbf{0} & \mathbf{A}-\mathbf{HC}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix} \triangleq \mathbf{F}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}, \tag{22}$$

which yields the characteristic equation

$$\Delta = \det((\lambda\mathbf{I} - \mathrm{F}) = \det(\lambda\mathbf{I} - (\mathbf{A\text{-}BK})\det(\lambda\mathbf{I} - (\mathbf{A\text{-}BK}), \tag{23}$$

whereby

$$\lambda(\mathbf{A} - \mathbf{BK}) \cup \lambda(\mathbf{A} - \mathbf{HC}), \tag{24}$$

where $\lambda(.)$ denotes the eigenvalues of (.). This property is the well-known *Separation Principle* for LTI systems [4, 11].

Further, (22) can be expressed in the $\{\mathbf{x}, \hat{\mathbf{x}}\}$ coordinates as

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\hat{\mathbf{x}}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}-\mathbf{BK} & \mathbf{BK}\\ \mathbf{0} & \mathbf{A}-\mathbf{BK}-\mathbf{HC}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \hat{\mathbf{x}}\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{H}\end{bmatrix}\mathbf{y}, \quad (25)$$

which is the closed-loop feedback control system and observer.

Computer simulation of (25) allows responses $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ to be compared. If the responses $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ are not satisfactory, for instance, $\hat{\mathbf{x}}(t)$ does not converge to $\mathbf{x}(t)$ quickly and smoothly, then the gain matrices $\mathbf{K}(t)$ and $\mathbf{H}(t)$ may be redesigned.

2.2 LTV Systems

Consider the LTV System

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}, \quad \mathbf{x}(0)=\mathbf{x}_o, \quad (26a)$$

$$\mathbf{y} = \mathbf{C}(t)\mathbf{x}, \quad (26b)$$

where $\mathbf{A}(t)$, $\mathbf{B}(t)$ and $\mathbf{C}(t)$ are time-varying matrices of compatible dimensions. Following the formulation in Section I above, we assume that the feedback control $\mathbf{u}(t)$ is given by

$$\mathbf{u} = -\mathbf{K}(t)\hat{\mathbf{x}}, \quad (27)$$

where $\mathbf{K}(t)$ is the time-varying feedback gain matrix and $\hat{\mathbf{x}}$ is an estimate of $\mathbf{x}$ generated by an observer. Recall that, for LTI systems, the pair [**A**, **B**] is controllable if and only if the pair [$\mathbf{A}^T$, $\mathbf{B}^T$] is observable [2, Theorem 6.5, Theorem of Duality]. However, for LTV systems, it follows that [$\mathbf{A}(t)$, $\mathbf{B}(t)$] is controllable if and only if $\left[-\mathbf{A}^T(t),\ \mathbf{B}^T(t)\right]$ is observable (see [2], Problem 6.22 and the Solution Manual for the proof).

An observer for (26) and (27) results in

$$\dot{\hat{\mathbf{x}}} = \boldsymbol{A}(t)\hat{\boldsymbol{x}} + \boldsymbol{B}(t)\boldsymbol{u} + \boldsymbol{H}(t)[\boldsymbol{y} - \boldsymbol{C}(t)\hat{\boldsymbol{x}}]$$

$$=\left[\mathbf{A}(t)-\mathbf{B}(t)\mathbf{K}(t)-\mathbf{H}(t)\mathbf{C}(t)\right]\hat{\mathbf{x}}+\mathbf{H}(t)\mathbf{y}, \quad \hat{\mathbf{x}}(t_o)=\hat{\mathbf{x}}_o. \quad (28)$$

Define the estimation error as

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}, \quad (29)$$

which yields, with (26) and (28),

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}$$

$$=\left[\mathbf{A}(t)-\mathbf{H}(t)\mathbf{C}((t)\right]e. \quad (30)$$

Combing (26) and (30), we obtain the augmented systems

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{e}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}(t)-\mathbf{B}(t)\mathbf{K}(t) & \mathbf{B}(t)\mathbf{K}(t)\\ \mathbf{0} & \mathbf{A}(t)-\mathbf{H}(t)\mathbf{C}(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}$$

$$\triangleq \begin{bmatrix}\boldsymbol{A}_{cl}(t) & \boldsymbol{B}(t)\boldsymbol{K}(t)\\ \mathbf{0} & \boldsymbol{A}_o(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{e}\end{bmatrix} \triangleq \boldsymbol{G}(t)\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{e}\end{bmatrix}, \quad (31)$$

The characteristic equation of (31) is given by

$$\Delta = \det(\left(\lambda(t)\mathbf{I}-\mathbf{G}(t)\right) = \det\left(\lambda(t)\mathbf{I}-\mathbf{A}_{cl}(t)\right)\ \det\left(\lambda(t)\mathbf{I}-\mathbf{A}_o(t)\right), \quad (32)$$

and yields the *Separation Principle* for LTV systems.

Equation (31) can also be expressed in the $\{\mathbf{x}, \hat{\mathbf{x}}\}$-coordinates as

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\hat{\mathbf{x}}}\end{bmatrix}=\begin{bmatrix}\mathbf{A}(t) & -\mathbf{B}(t)\mathbf{K}(t)\\ \mathbf{0} & \mathbf{A}t)-\mathbf{B}(t)\mathbf{K}(t)-\mathbf{H}(t)\mathbf{C}(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \hat{\mathbf{x}}\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{H}(t)\end{bmatrix}\mathbf{y}. \quad (33)$$

Similar to (25) for LTI systems, (33) is needed for implementing LTV closed-loop feedback control systems and observers. Simulation responses $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ of (33) can be used to adjust the gain matrices $\mathbf{K}(t)$ and $\mathbf{H}(t)$ to improve the performance of $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$.

A block diagram for the LTV feedback control system and LTV observer is shown in Figure 1.

F6:Controllability and Observability of LTV Systems

Consider the LTV system described by (1), repeated here for ease of reference:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}+\mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_o)=\mathbf{x}_o, \quad t_o\in[a,b], \quad (1a)$$

$$\mathbf{y} = \mathbf{C}(t)\mathbf{x}. \quad (1b)$$

We have the following sufficient conditions for the controllability and observability of LTV systems.

Theorem 1: Controllability of LTV systems (2, Th 6.12; 6, [12], Th 2.5; 16)

Let $\mathbf{A}(t)$ and $\mathbf{B}(t)$ be (n-1) times continuously differentiable. Then the pair [$\mathbf{A}(t)$, $\mathbf{B}(t)$] is controllable at t_o if there exists a finite $t_1 > t_o$ such that

$$\text{rank}[\mathbf{M}(t)] = n, \quad (34)$$

where

$$\mathbf{M}(t) = \left[\mathbf{M}_o(t) \mid \mathbf{M}_1(t) \mid \cdots \mid \mathbf{M}_{n-1}(t)\right], \quad (35a)$$

$$\mathbf{M}_o(t) = \mathbf{B}(t), \quad (35b)$$

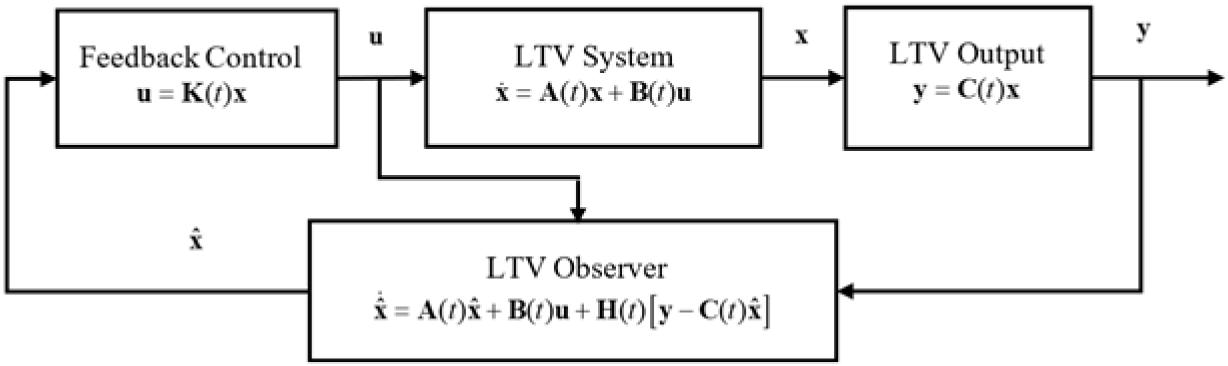

Figure 1. Block diagram for LTV feedback control system with LTV observer

$$\mathbf{M}_{m+1}(t) = -\mathbf{A}(t)\mathbf{M}_m(t) + \dot{\mathbf{M}}_m(t), \tag{35c}$$

for $m = 0, 1, \ldots, n-1$. ■

For LTI systems, (35) yields

$$\mathbf{M} = \left[\mathbf{B} \mid -\mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid -\mathbf{A}^3\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right], \tag{36}$$

which has the same rank as the standard controllability theorem of the pair [A, **B**] given by

$$\bar{\mathbf{M}} = \left[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid \mathbf{A}^3\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right], \tag{37}$$

i.e., the alternate minus sign in (36) do not affect its rank and $\text{rank}(\bar{\mathbf{M}}) = \text{rank}(\mathbf{M})$.

Theorem 2: Observability of LTV systems [2, Theorem 6.012]

Let **A(**t) and **C**(t) be (n-1) times continuously differentiable. Then the pair [**A**(t), **C**(t)] is observable at t_o if there exists a finite $t_1 > t_o$ such that

$$\text{rank}[\mathbf{N}(t)] = n, \tag{38}$$

where

$$\mathbf{N}(t) = \begin{bmatrix} \mathbf{N}_o(t) \\ \mathbf{N}_1(t) \\ \vdots \\ \mathbf{N}_{n-1}(t) \end{bmatrix}, \tag{39a}$$

$$\mathbf{N}_o(t) = \mathbf{C}(t), \tag{39b}$$

$$\mathbf{N}_{m+1}(t) = \mathbf{N}_m(t)\mathbf{A}(t) + \dot{\mathbf{N}}(t), \tag{39c}$$

for $m = 0, 1, \ldots, n-1$. ■

Theorem 3: Relationship between Controllability and Observability of LTV systems [2]

The pair [**A**(t), **B**(t)] is controllable at t_o if and only if [-$\mathbf{A}^{\mathbf{T}}(t)$, $\mathbf{B}^{\mathbf{T}}(t)$] is observable at t_o.

Proof: The proof can be found in [2], Solution Manual for Problem 6.22. ■

3 Analytical Solutions and Simulations of LTV Systems, and Design of LTV Feedback Control Systems and LTV Observers

The analytical solutions of the fundamental and state transition matrices of LTV systems and the design of feedback control and observers will be investigated in this section. Matlab solutions and simulations will be given as well.

Example 1: Second-order LTV system [6, 17]

Consider the LTV system:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} = \begin{bmatrix} -6t^2 & 3t^5 \\ 0 & -3t^2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(t_o) = \mathbf{x}_o, \tag{1-1}$$

where **A**(t) is an upper triangular matrix. The primary objectives of this example are:

(1a): Solving for **x**(t), the fundamental matrix **X**(t), and the state transition matrix $\mathbf{\Phi}(t, 0)$;

(1b): Simulating the responses of **x**(t) in Matlab with normalized ICs: $x_1(0) = 1$ and $x_2(0) = 1$.

Solution:

(1a): First, we check the matrix commutativity properties of **A**(t):

$$\mathbf{A}(t_1)\mathbf{A}(t_2) \neq \mathbf{A}(t_2)\mathbf{A}(t_1) \quad \text{and} \quad \mathbf{A}(t)\mathbf{\Psi}(t) \neq \mathbf{\Psi}(t)\mathbf{A}(t), \tag{1-2}$$

where $\mathbf{\Psi}(t)$ denotes the integral of $\mathbf{A}(t)$, i.e.,

$$\mathbf{\Psi}(t) = \int_{t_o}^{t} \mathbf{A}(\tau)d\tau \, . \qquad (1\text{-}3)$$

Since both conditions are not met, then $\mathbf{X}(t) \neq \exp\left(\int_{t_o}^{t} \mathbf{A}(\tau)d\right)$ and $\dot{\mathbf{X}}(t) \neq \mathbf{A}(t)\mathbf{X}(t)$. The problem has been investigated in [6] and [17]. The solutions of Wu and Jain are listed below, respectively:

(i) Wu: $\mathbf{\Phi}_{Wu}(t,0) = \begin{bmatrix} e^{-2t^3} & e^{-2t^3} - e^{-t^3} + t^3 e^{-t^3}) \\ 0 & e^{-t^3} \end{bmatrix}, \ \mathbf{\Phi}_{Wu}(0,0) = \mathbf{I}_2$, (1-4)

$$\Rightarrow \quad \mathbf{x}(t)_{Wu} = \begin{bmatrix} x_1(t)_{Wu} \\ x_2(t)_{Wu} \end{bmatrix} = \begin{cases} e^{-2t^3} x_1(0) + (e^{-2t^3} - e^{-t^3} + t^3 e^{-t^3}) x_2(0), \\ e^{-t^3} x_2(0). \end{cases} \qquad (1\text{-}5)$$

(ii) Jain:

$$\mathbf{\Phi}_{Jain}(t,0) = \begin{bmatrix} e^{-2t^3} & \frac{t^3}{2}\left(e^{-t^3} - e^{-2t^3}\right) \\ 0 & e^{-t^3} \end{bmatrix}, \ \mathbf{\Phi}_{Jain}(0,0) = \mathbf{I}_2, \quad (1\text{-}6)$$

$$\Rightarrow \mathbf{x}(t)_{Jain} = \begin{bmatrix} x_1(t)_{Jain} \\ x_2(t)_{Jain} \end{bmatrix} = \begin{cases} e^{-2t^3} x_1(0) + \frac{t^3}{2}\left(e^{-t^3} - e^{-2t^3}\right) x_2(0), \\ e^{-t^3} x_2(0), \end{cases} \qquad (1\text{-}7)$$

where the state transition matrices $\mathbf{\Phi}_{Wu}(t,0) \neq \mathbf{\Phi}_{Jain}(t,0)$. It follows that

$$\frac{\partial \mathbf{\Phi}_{Wu}(t,0)}{\partial t} = \mathbf{A}(t)\mathbf{\Phi}_{Wu}(t,0) \quad \Rightarrow \quad \text{Wu's solution is correct,} \qquad (1\text{-}8)$$

$$\frac{\partial \mathbf{\Phi}_{Jain}(t,0)}{\partial t} \neq \mathbf{A}(t)\mathbf{\Phi}_{Jain}(t,0) \quad \Rightarrow \quad \text{Jain's solution is incorrect.} \qquad (1\text{-}9)$$

If we check more closely, it follows from (1-5) and (1-7) that

$$x_2(t)_{WU} = x_2(t)_{Jain}, \qquad (1\text{-}10)$$

but the error between $x_1(t)_{Wu}$ and $x_1(t)_{Jain}$ is given by

$$e_1(t) = x_1(t)_{Wu} - x_1(t)_{Jain}$$

$$= \left[e^{-t^3}\left(\frac{t^3}{2} - 1\right) + e^{-2t^3}\left(\frac{t^3}{2} + 1\right) \right] x_2(0) \quad \to 0 \quad \text{as } t \to \infty \, . \qquad (1\text{-}11)$$

Further, the error between $\mathbf{\Phi}_{Wu}(t,0)$ and $\mathbf{\Phi}_{Jain}(t,0)$ is given by

$$\mathbf{E}(t) = \mathbf{\Phi}_{Wu}(t,0) - \mathbf{\Phi}_{Jain}(t,0) = \begin{bmatrix} 0 & e^{-2t^2} - e^{-t^3} + \frac{1}{2} t^3 \left(e^{-t^3} + e^{-2t^3}\right) \\ 0 & 0 \end{bmatrix}$$

$\to 0$ as $t \to \infty$. (1-12)

The Matlab program using Matlab's **EXPM** command that yields (1-6) is listed below:

```
syms t

A = [-6*t^2   3*t^5; 0   -3*t^2];

M = int(A,t) = [-2*t^3, 1/2*t^6]
               [      0,      -t^3];

Xmatlab= simplify(expm(M)) =

= [ exp(-2*t^3),   -1/2*t^3*exp(-t^3)*(-1+exp(-t^3))]
  [            0,                            exp(-t^3)],
```

which is identical to (1-6) and is incorrect.

(1b): The responses of the analytical solutions $\mathbf{x}(t)_{Wu}$ and $\mathbf{x}(t)_{Jain}$ given by (1-5) and (1-7), respectively, are plotted in Figure 2. For comparisons, simulations of the time-varying ODE (1-1) using Matlab's ODE45 are also plotted. All the plots in Figure 2 agree with the observations given by (1-10) and (1-11). Hence, based on the matrix differential equations $\frac{\partial \mathbf{\Phi}_{Wu}(t,0)}{\partial t} = \mathbf{A}(t)\mathbf{\Phi}_{Wu}(t,0)$ and $\frac{\partial \mathbf{\Phi}_{Jain}(t,0)}{\partial t} \neq \mathbf{A}(t)\mathbf{\Phi}_{Jain}(t,0)$ given by (1-8) and (1-9), the error equation given by (1-11), and all the simulation results, we conclude that Wu's method yields the correct solution to $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ and Jain's solutions are incorrect. Recall that Matlab will yield correct solutions if $\mathbf{A}(t)$ given in (1-1) is a commutative matrix.

Example 2: Design of Feedback Control for LTV System and LTV observer with Prescribed Properties

Consider the LTV system described by **[6]**

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)u = \begin{bmatrix} 0 & -1-\exp(-t) \\ 1 & -\exp(-t) \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \, , \qquad (2\text{-}1a)$$

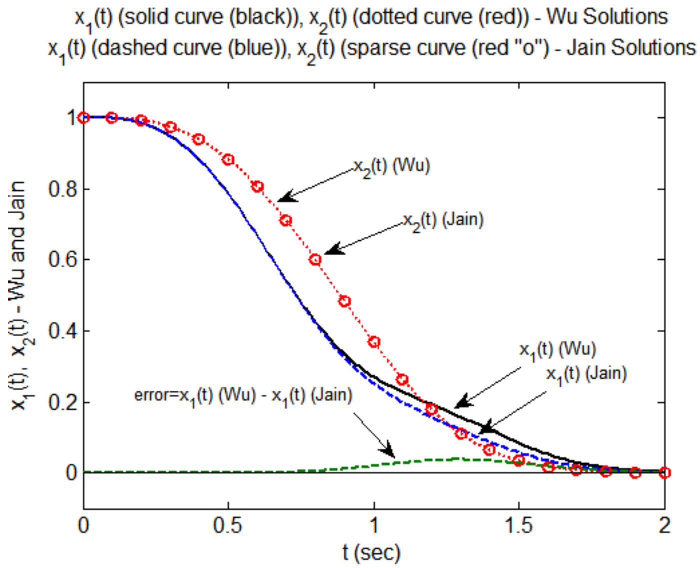

Figure 2: Responses of $\mathbf{x}_{Wu}(t)$ given by (1-4) and $\mathbf{x}_{Jain}(t)$ given by (1-6) with $x_1(0)=1$ and $x_2(0)=1$, and the error $e_1(t)$ given by (1-11). All the solutions decay to zero as $t\to\infty$. ■

$$\mathbf{y}=\mathbf{C}(t)\mathbf{x}=[0\ 1]\mathbf{x}=x_2. \tag{2-1b}$$

Do the following:

(2a): Design a feedback control system for (2-1) such that the resulting closed-loop system matrix

$$\mathbf{A}_{cl}(t)=\mathbf{A}(t)-\mathbf{B}(t)\mathbf{K}(t) \tag{2-2}$$

is triangular, commutative, and nilpotent, where $\mathbf{K}(t)$ is the feedback gain matrix. Determine the analytical solutions for the fundamental matrix $\mathbf{X}_{cl}(t)$ and state transition matrix $\mathbf{\Phi}_{cl}(t,0)$ associated with $\mathbf{A}_{cl}(t)$ given by (2-2).

(2b): Design an observer for (2-1) such that the observer matrix

$$\mathbf{A}_o(t)=\mathbf{A}(t)-\mathbf{H}(t)\mathbf{C}(t) \tag{2-3}$$

is triangular, commutative, and nilpotent, where $\mathbf{H}(t)$ is the observer gain matrix. Determine the analytical solutions for the fundamental matrix $\mathbf{X}_o(t)$ and state transition matrix $\mathbf{\Phi}_o(t,0)$ associated with $\mathbf{A}_o(t)$ given by (2-3).

(2c): Simulate and plot the responses of $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$.

Solution:

(2a) Design of LTV Control System

Solution:

Step 1: Check the controllability matrix of the pair $[\mathbf{A}(t), \mathbf{B}(t)]$ given by (36), Theorem 1. We obtain:

$$\mathbf{M}(t)=[\mathbf{M}_o(t)\ \vdots\ \mathbf{M}_1(t)]$$

$$=[\mathbf{M}_o(t)\ \vdots\ -\mathbf{A}(t)\mathbf{M}_o(t)+\dot{\mathbf{M}}_o(t)]=[\mathbf{B}(t)\ \vdots\ -\mathbf{A}(t)\mathbf{B}(t)]$$

$$=\begin{bmatrix}0 & 1+\exp(-t)\\ 1 & \exp(-t)\end{bmatrix}, \tag{2-4a}$$

where

$$\mathbf{M}_o(t)=\mathbf{B}(t), \tag{2-4b}$$

$$\mathbf{M}_1(t)=-\mathrm{A}(t)*\mathbf{M}\mathrm{o}(t)=\begin{bmatrix}1+\exp(-t)\\ \exp(-t)\end{bmatrix}. \tag{2-4c}$$

The determinant $\det(\mathbf{M}(t))$ of $\mathbf{M}(t)$ is given by

$$\det(\mathbf{M}(t))=-1-\exp(-t)\neq 0 \text{ for all } t, \tag{2-5}$$

$\Rightarrow$ rank$[\mathbf{M}(t)]$ = 2 for all t $\Rightarrow$ LTV system (2.1) is controllable for all t.

Step 2: Design of feedback control system

Let a feedback control be given by

$$u=-\mathbf{K}(t)\hat{\mathbf{x}}, \tag{2-6}$$

where $\mathbf{K}(t)=[k_1(t)\quad k_2(t)]$ is a feedback gain matrix, and $\hat{\mathbf{x}}$ is an estimate generated by an observer . Substituting (2-6) into (2-1) yields the closed-loop control system

$$\dot{\mathbf{x}}=\mathbf{A}(t)\mathbf{x}-\mathbf{B}(t)\mathbf{K}(t)\hat{\mathbf{x}}. \tag{2-7}$$

Since $[\mathbf{A}(t), \mathbf{B}(t)]$ is a controllable pair, a suitable control gain matrix $\mathbf{K}(t)$ exists such that $\mathbf{A}_{cl}(t)$ has desirable properties, specifically, $\mathbf{A}_{cl}(t)$ being triangular and commutative.

Step 3: Determine the control gain matrix $\mathbf{K}(t)$ in (2-6) and (2-7). We have

$$\mathbf{A}_{cl}(t)=\mathbf{A}(t)-\mathbf{B}(t)\mathbf{K}(t)=\begin{bmatrix}0 & -1-\exp(-t)\\ 1-k_1(t) & -\exp(-t)-k_2(t)\end{bmatrix}. \tag{2-8}$$

Setting $k_1(t)=1$ and $k_2(t)=-\exp(-t)$ in (2-8) yields

$$\mathbf{A}_{cl}(t)=\mathbf{A}(t)-\mathbf{B}(t)\mathbf{K}(t)=\begin{bmatrix}0 & -1-\exp(-t)\\ 0 & 0\end{bmatrix}, \quad (2\text{-}9)$$

which is a commutative and triangular matrix with zero diagonal elements so that the design criteria are satisfied.

Step 4: Determination of the fundamental matrix $\mathbf{X}_{cl}(t)$ and transition matrix $\mathbf{\Phi}_{cl}(t,0)$.

Since $\mathbf{A}_{cl}(t)$ is a triangular and commutative matrix**,** its associated fundamental matrix $\mathbf{X}_{cl}(t)$ and state transition matrix $\mathbf{\Phi}_{cl}(t,0)$ can be determined readily as follows: We have

$$\int_o^t \mathbf{A}_{cl}(\tau)d\tau = \begin{bmatrix}0 & -t+\exp(-t)\\ 0 & 0\end{bmatrix}, \quad \left(\int_o^t \mathbf{A}_{cl}(\tau)d\tau\right)^2 = \mathbf{0}. \quad (2\text{-}10)$$

Since $\left(\int_o^t \mathbf{A}_{cl}(\tau)d\tau\right)^2 = \mathbf{0}$, $\mathbf{X}_{cl}(t)$ is given by

$$\mathbf{X}_{cl}(t) = \mathrm{e}^{\int_o^t \mathbf{A}_{cl}(\tau)d\tau} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_o^t \mathbf{A}_{cl}(\tau)d\tau\right)^k = \mathbf{I}_2 + \int_o^t \mathbf{A}_{cl}(\tau)d\tau, \quad (2\text{-}11)$$

which yields a finite-sum solution

$$\mathbf{X}_{cl}(t) = \mathbf{I}_2 + \int_o^t \mathbf{A}_{cl}(\tau)d\tau = \begin{bmatrix}1 & -t+\mathrm{e}^{-t}\\ 0 & 1\end{bmatrix}, \quad \mathbf{X}_{cl}(0) = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}. \quad (2\text{-}12)$$

Using (2-12), the state transition matrix $\mathbf{\Phi}_{cl}(t,0)$ is obtained as

$$\mathbf{\Phi}_{cl}(t,0) = \mathbf{X}_{cl}(t)\mathbf{X}_{cl}^{-1}(0) = \begin{bmatrix}1 & -1-t+e^{-t}\\ 0 & 1\end{bmatrix}, \quad \mathbf{\Phi}_{cl}(t,0) = \mathbf{I}_2. \quad (2\text{-}13)$$

Equations (2-12) and (2-13) satisfy, respectively, the matrix differential equations:

$$\dot{\mathbf{X}}_{cl}(t) = \mathbf{A}_{cl}(t)\mathbf{X}_{cl}(t), \; \mathbf{X}_{cl}(0), \quad (2\text{-}14)$$

$$\dot{\mathbf{\Phi}}_{cl}(t,0) = \mathbf{A}_{cl}(t)\mathbf{\Phi}_{cl}(t,0), \; \mathbf{\Phi}_{cl}(0,0) = \mathbf{I}_2, \quad (2\text{-}15)$$

which confirm that $\mathbf{X}_{cl}(t)$ and $\mathbf{\Phi}_{cl}(t,0)$ solve (2-14) and (2-15), respectively.

(2c): Design of LTV Observer with Prescribed Properties

Step 1: Check the observability matrix of the pair [$\mathbf{A}(t)$, $\mathbf{C}(t)$] given by (40), Theorem 2. We obtain:

$$\mathbf{N}(t) = \begin{bmatrix}\mathbf{N}_o(t)\\ \mathbf{N}_1(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & -\exp(-t)\end{bmatrix}, \quad (2\text{-}16)$$

The determinant det($\mathbf{N}(t)$) of $\mathbf{N}(t)$ is given by

$$\det(\mathbf{N}(t)) = -1, \quad (2\text{-}17)$$

$\Rightarrow$ rank[$\mathbf{N}(t)$] = 2 for all t $\Rightarrow$ LTV system (2.1) is observable for all t.

Step 2: An observer for (2-1) has been designed in [8, Eq. (3.215)] and is given by,

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}(t)\hat{\mathbf{x}} + \mathbf{H}(t)[\mathbf{y} - \mathbf{C}(t)\hat{\mathbf{x}}] + \mathbf{B}(t)\mathbf{u}\\ &= [\boldsymbol{A}(t) - \boldsymbol{H}(t)\boldsymbol{C}(t)]\hat{\boldsymbol{x}} + \boldsymbol{H}(t)\boldsymbol{y} + \boldsymbol{B}(t)\boldsymbol{u}\\ &\triangleq \boldsymbol{A}_o(t)\hat{\boldsymbol{x}} + \boldsymbol{H}(t)\boldsymbol{y} + \boldsymbol{B}(t)\boldsymbol{u}, \end{aligned} \quad (2\text{-}18)$$

where the observer gain matrix $\mathbf{H}(t)$ is given by

$$\mathbf{H}(t) = \begin{bmatrix}h_1(t)\\ h_2(t)\end{bmatrix}. \quad (2\text{-}19)$$

Substituting (2-19) into (2-18) yields

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix}0 & -1-e^{-t}-h_1(t)\\ 1 & -e^{-t}-h_2(t)\end{bmatrix}\hat{\mathbf{x}} + \mathbf{H}(t)\mathbf{y} + \mathbf{B}(t)\mathbf{u}. \quad (2\text{-}20)$$

In [6], the observer gain matrix $\mathbf{H}(t)$ was chosen as

$$\mathbf{H}(t) = \begin{bmatrix}h_1(t)\\ h_2(t)\end{bmatrix} = \begin{bmatrix}m_o - 1 - \mathrm{e}^{-t}\\ m_1 - \mathrm{e}^{-t}\end{bmatrix}, \quad (2\text{-}21)$$

where m_o and m_1 are constants. Substituting (2-21) into (2-20) yields an LTI observer

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix}0 & -m_o\\ 1 & -m_1\end{bmatrix}\hat{\mathbf{x}} + \mathbf{H}(t)\mathbf{y} + \mathbf{B}(t)\mathbf{u} \triangleq \overline{\mathbf{A}}_o\hat{\mathbf{x}} + \mathbf{H}(t)\mathbf{y} + \mathbf{B}(t)\mathbf{u}. \quad (2\text{-}22)$$

where $\overline{\mathbf{A}}_o$ is a constant matrix. The eigenvalues of $\overline{\mathbf{A}}_o$ are given by $\lambda = \dfrac{-m_1 \pm \sqrt{m^2 + 4m_o}}{2}$ which can be used to guide the choice of m_o and m_1 for stability analysis.

Simulation studies of the LTV closed-loop control and observer can be obtained by using (2-1) and (2-22), repeated for ease of reference,

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)u = \begin{bmatrix}0 & -1-\exp(-t)\\ 1 & -\exp(-t)\end{bmatrix}\mathbf{x} + \begin{bmatrix}0\\ 1\end{bmatrix}u,$$

$$\mathbf{y}=\mathbf{C}(t)\boldsymbol{x}=[0\ 1]\mathbf{x}=x_2,\quad \mathbf{x}(0))=\mathbf{x}_o; \tag{2-1}$$

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} 0 & -m_o \\ 1 & -m_1 \end{bmatrix} \hat{\mathbf{x}} + \mathbf{H}(t)\mathbf{y} + \mathbf{B}(t)\mathbf{u}, \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_o \tag{2-22}$$

where

$$\underline{u} = -\mathbf{K}(t)\,\hat{\mathbf{x}} = \begin{bmatrix} 1 & -\exp(-t) \end{bmatrix} \hat{\mathbf{x}}. \tag{2-23}$$

The responses of $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ are as shown in Figure 3, where $\mathbf{x}(t)$ converges to $\hat{\mathbf{x}}(t)$ quickly and smoothly.

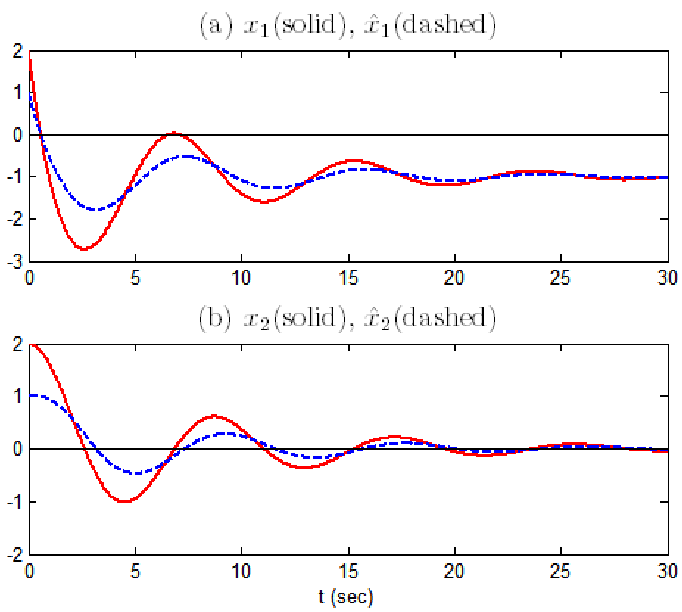

Figure 3: Responses of the estimate $\hat{\mathbf{x}}(t)$ given by (2-22) converges to $\mathbf{x}(t)$ given by (2-1) quickly and smoothly for m_o=1 and m_1=1. The initial conditions were both chosen as $\mathbf{x}(0)) = \hat{\mathbf{x}}(0) = [2\ \ 1]^T$ ■

Example 3: Design of Feedback Control for LTV System and LTV observer with prescribed Properties

Consider a 4th-order LTV system described by the homogenous equation

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2t^3 & 3t \\ 1 & 0 & 0 & 0 \\ 3t^2 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}. \tag{3-1}$$

It is an LTV system investigated in [10] that has interesting properties, for example,

$$\mathbf{A}(t)\mathbf{A}(s) \neq \mathbf{A}(s)\mathbf{A}(t), \tag{3-2a}$$

but

$$\mathbf{A}(t)\int_o^t \mathbf{A}(\tau)d\tau = \int_o^t \mathbf{A}(\tau)d\tau\,\mathbf{A}(t) \quad \Rightarrow \quad \left[\mathbf{A}(t),\ \int_o^t \mathbf{A}(\tau)d\tau\right] = \mathbf{0}. \tag{3-2b}$$

(3a): Design a feedback control system for (3-1) such that the resulting closed-loop system matrix

$$\mathbf{A}_{cl}(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t) \tag{3-3}$$

is triangular and commutative, where $\mathbf{K}(t)=\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$, is the feedback gain matrix. Determine $\mathbf{K}(t)$ and the analytical solutions for the fundamental matrix $\mathbf{X}_{cl}(t)$ and state transition matrix $\mathbf{\Phi}_{cl}(t,0)$ associated with $\mathbf{A}_{cl}(t)$ given by (3-3).

(3b): Design an observer for (3-1) such that the observer matrix

$$\mathbf{A_o}(\mathrm{t}) = \mathbf{A}(t) - \mathbf{H}(t)\mathbf{C}(t) \tag{3-4}$$

is triangular and commutative, where $\mathbf{H}(t)=\begin{bmatrix} h_1 & h_2 & h_3 & h_4 \end{bmatrix}^T$ is the observer gain matrix. Determine $\mathbf{H}(t)$ and the analytical solutions for the fundamental matrix $\mathbf{X}_o(t)$ and state transition matrix $\mathbf{\Phi}_o(t,0)$ associated with $\mathbf{A}_o(t)$ given by (3-4).

Solutions:

(3a): Design of LTV feedback control system with prescribed properties

Since (3-1) is a system with no input, we need to modify it for feedback control analysis as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}, \tag{3-5}$$

where $\mathbf{B}(t)$ is to be chosen such that the rows of $\mathrm{B}(t)\mathbf{K}(t)$ can modify the rows of $\mathbf{A}(t)$ to mach the design criteria, in particular $\mathbf{A}_{cl}(t)$ being a triangular and commutative matrix.

Design algorithm:

Step 1: Examining the structure of $\mathbf{A}(t)$, it follows that an upper triangular and commutative matrix $\mathbf{A}_{cl}(t)$ can be obtained by replacing the terms $2t^3$ and $3t$ in the second row of $\mathbf{A}(t)$ by an 0 (zero). This suggests that $\mathbf{B}(t)$ should be chosen with a nonzero row and has the form

$$\mathbf{B}(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{3-6}$$

Substituting (3-6) into (3-3) yields

$$\mathbf{A}_{cl}(t)=\mathbf{A}(t)-\mathbf{B}(t)\mathbf{K}(t)=\begin{bmatrix}0 & 0 & 0 & 0\\ -k_1 & -k_2 & 2t^3-k_3 & 3t-k_4\\ 1 & 0 & 0 & 0\\ 3*t^2 & 0 & 0 & 0\end{bmatrix}. \quad (3\text{-}7)$$

Setting k_1=0, k_2=0, k_3= $2t^3$ and k_4=$3t$ into (3-7) yields

$$\mathbf{A}_{cl}(t)=\mathbf{A}(t)-\mathbf{B}(t)\mathbf{K}(t)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 3*t^2 & 0 & 0 & 0\end{bmatrix}, \quad (3\text{-}8)$$

which is a lower triangular and commutative matrix. It is emphasize that $\mathbf{A}_{cl}(t)$ given by (3-8) was obtained without regard to whether the pair [$\mathbf{A}(t)$, $\mathbf{B}(t$] is controllable or uncontrollable.

Step 2: Check the controllability matrix $\mathbf{M}(t)$ of the pair [$\mathbf{A}(t)$, $\mathbf{B}(t)$] given by (36), Theorem 1 with $\mathbf{B}(t)$ given by (3-6). We have:

$$\mathbf{M}(t)=[\mathbf{M}_o(t) \quad \mathbf{M}_1(t) \quad \mathbf{M}_2(t) \quad \mathbf{M}_3(t)], \quad (3\text{-}9a)$$

where

$$\boldsymbol{M}_{\mathrm{o}}(t)=\mathbf{B}(t), \quad (3\text{-}9b)$$

$$\boldsymbol{M}_1(t)=-\mathbf{A}(t)\mathbf{M}_o(t)+\dot{\mathbf{M}}_o(t)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\quad \mathbf{M}_2(t)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},$$

$$M_3(t)=-\mathbf{A}(t)\mathbf{M}_2(t)+\dot{\mathbf{M}}_2(t)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}. \quad (3\text{-}9c)$$

which yields

$$\mathbf{M}(t)=\begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \quad (3\text{-}9d)$$

The determinant det($\mathbf{M}(t)$) of $\mathbf{M}(t)$) is given by

$$\det(\mathbf{M}(t))=0 \quad \Rightarrow \quad \text{LTV system is uncontrollable.} \quad (3\text{-}10)$$

Hence, there exists no feedback gain matrix $\mathbf{K}(t)$ in (3-3) that can change the structure of $\mathbf{A}(t)$ to that of $\mathbf{A}_{cl}(t)$. However, the method proposed in the paper has just accomplished the design objective as shown by (3-8). On the other hand, given a general $\mathbf{A}(t)$, the proposed method may fail.

Step 3: Determination of the fundamental matrix $\mathbf{X}_{cl}(t)$ and transition matrix $\mathbf{\Phi}_{cl}(t,0)$.

Since $\mathbf{A}_{cl}(t)$ given by (3-8) is a triangular and commutative matrix**,** its associated fundamental matrix $\mathbf{X}_{cl}(t)$ and state transition matrix $\mathbf{\Phi}_{cl}(t,0)$ can be determined readily as follows. We have

$$\int_o^t \mathbf{A}_{cl}(\tau)d\tau=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ t & 0 & 0 & 0\\ t^2 & 0 & 0 & 0\end{bmatrix} \text{ and } \left(\int_o^t \mathbf{A}_{cl}(\tau)d\tau\right)^2=\mathbf{0}. \quad (3\text{-}11)$$

Since $\left(\int_o^t \mathbf{A}_{cl}(\tau)d\tau\right)^2=\mathbf{0}$, $\mathbf{X}_{cl}(t)$ is given by

$$\mathbf{X}_{cl}(t)=\mathrm{e}^{\int_o^t \mathbf{A}_{cl}(\tau)d\tau}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\int_o^t \mathbf{A}_{cl}(\tau)d\tau\right)^k=\mathbf{I}_4+\int_o^t \mathbf{A}_{cl}(\tau)d\tau, \quad (3\text{-}12)$$

which yields the finite-sum solution

$$\mathbf{X}_{cl}(t)=\mathbf{I}_4+\int_o^t \mathbf{A}_{cl}(\tau)d\tau=\mathrm{X}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ t & 0 & 1 & 0\\ t^3 & 0 & 0 & 1\end{bmatrix},\quad \mathbf{X}_{cl}(0)=\mathbf{I}_4. \quad (3\text{-}13)$$

Using (3-13), the state transition matrix $\mathbf{\Phi}_{cl}(t,0)$ is obtained as

$$\mathbf{\Phi}_{cl}(t,0)=\mathbf{X}_{cl}(t)\mathbf{X}_{cl}^{-1}(0)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ t & 0 & 1 & 0\\ t^3 & 0 & 0 & 1\end{bmatrix},\quad \mathbf{\Phi}_{cl}(t,0)=\mathbf{I}_4. \quad (3\text{-}14)$$

Equations (3-13) and (3-14) satisfy, respectively, the matrix differential equations:

$$\dot{\mathbf{X}}_{cl}(t)=\mathbf{A}_{cl}(t)\mathbf{X}_{cl}(t),\ \ \mathbf{X}_{cl}(0), \quad (3\text{-}15)$$

$$\dot{\mathbf{\Phi}}_{cl}(t,0)=\mathbf{A}_{cl}(t)\mathbf{\Phi}_{cl}(t,0),\ \ \mathbf{\Phi}_{cl}(0,0)=\mathbf{I}_4, \quad (3\text{-}16)$$

which confirm that $\mathbf{X}_{cl}(t)$ and $\mathbf{\Phi}_{cl}(t,0)$ solve (3-15) and (3-16), respectively.

(3b): Design of LTV Observer with Prescribed Properties

Since (3-1) is a system with no output, we need to modify it for observer design as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{y} = \mathbf{C}(t)\mathbf{x}, \qquad (3\text{-}17)$$

where $\mathbf{C}(t)$ is to be chosen such that $\mathbf{H}(t)\mathbf{C}(t)$ satisfies the design criteria. An LTV observer for (3-17) can be designed as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}(t)\hat{\mathbf{x}} + \mathbf{H}(t)[\mathbf{y} - \mathbf{H}(t)C(t)\hat{\mathbf{x}}]$$

$$= [\mathbf{A}(t) - \mathbf{H}(t)\mathbf{C}(t)]\hat{\mathbf{x}} + \mathbf{H}(t)\mathbf{y} = \mathbf{A}_o(t)\hat{\mathbf{x}} + \mathbf{H}(t)\mathbf{y}, \qquad (3\text{-}18)$$

where $\mathbf{H}(t)$ is the observer gain matrix given by

$$\mathbf{H}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \\ h_4(t) \end{bmatrix}. \qquad (3\text{-}19)$$

Design Algorithm

Step 1: Once again, examining the structure of $\mathbf{A}(t)$, it follows that a lower triangular and commutative matrix $\mathbf{A}_o(t)$ can be obtained by replacing the third and fourth elements in the first column of $\mathbf{A}(t)$. This suggests that $\mathbf{C}(t)$ can be chosen as

$$\mathbf{C}(t) = [1 \quad 0 \quad 0 \quad 0]. \qquad (3\text{-}20)$$

Substituting (3-20) into (3-18) yields

$$\mathbf{A}_o(t) = \begin{bmatrix} -h_1 & 0 & 0 & 0 \\ -h_2 & 0 & 2t^3 & 3t \\ 1-h_3 & 0 & 0 & 0 \\ 3t^2 - h_4(t) & 0 & 0 & 0 \end{bmatrix}. \qquad (3\text{-}21)$$

Setting $h_1=0$, $h_2=0$, $h_3=1$ and $h_4=3t^2$ yields

$$\mathbf{A}_o(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2t^3 & 3t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (3\text{-}22)$$

which is an upper triangular and commutative matrix.

Step 2: Check the observability matrix $N(t)$ of the pair $[\mathbf{A}(t), \mathbf{C}(t)]$ given by (40), Theorem 2, for $n=4$. We have:

$$\mathbf{N}(t) = [\mathbf{N}_o(t) \quad \mathbf{N}_1(t) \quad \mathbf{N}_2(t) \quad N_3(t)], \qquad (3\text{-}23a)$$

where

$$\mathbf{N}_o(t) = \mathbf{C}(t), \qquad (3\text{-}23b)$$

$$N_1(t) = \mathbf{N}_o(t)\mathbf{A}(t) + \dot{\mathbf{N}}_o(t) = \mathbf{0}, \; \mathbf{N}_2(t) = \mathbf{N}_1(t)\mathbf{A}(t) + \dot{\mathbf{N}}_1(t) = \mathbf{0},$$

$$\mathbf{N}_3(t) = \mathbf{N}_2\mathbf{A}(t) + \dot{\mathbf{N}}_2(t) = \mathbf{0}, \qquad (3\text{-}23c)$$

which yields

$$\mathbf{N}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (3\text{-}23d)$$

The determinant $\det(\mathbf{N}(t))$ of $\mathbf{N}(t))$ is given by

$$\det(\mathbf{N}(t)) = 0 \quad \Rightarrow \quad \text{LTV system is uncontrollable.} \qquad (3\text{-}24)$$

Hence, there exists no observer gain $\mathbf{H}(t)$ that can change the structure of A(t) to that of $\mathbf{A}_{cl}(t)$. But the method proposed in this paper has just accomplished the design objective as shown by (3-22). On the other hand, given a general $\mathbf{A}(t)$, the proposed method may not work.

Step 3: Determination of the fundamental matrix $\mathbf{X}_o(t)$ and transition matrix $\mathbf{\Phi}_o(t,0)$.

Since $\mathbf{A}_o(t)$ given by (3-22) is a triangular and commutative matrix, its associated fundamental matrix $\mathbf{X}_o(t)$ and state transition matrix $\mathbf{\Phi}_o(t,0)$ can be determined readily as follows: We have

$$\int_o^t \mathbf{A}_o(\tau)d\tau = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5t^4 & 1.5t^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \left(\int_o^t \mathbf{A}_o(\tau)d\tau\right)^2 = \mathbf{0}. \qquad (3\text{-}25)$$

Since $\left(\int_o^t \mathbf{A}_o(\tau)d\tau\right)^2 = \mathbf{0}$, $\mathbf{X}_o(t)$ is given by

$$\mathbf{X}_o(t) = e^{\int_o^t \mathbf{A}_o(\tau)d\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_o^t \mathbf{A}_o(\tau)d\tau\right)^k = \mathbf{I}_4 + \int_o^t \mathbf{A}_o(\tau)d\tau, \qquad (3\text{-}26)$$

which yields a finite-sum solution

$$\mathbf{X}_{cl}(t) = \mathbf{I}_4 + \int_o^t \mathbf{A}_{cl}(\tau)d\tau = \text{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0.5t^4 & 1.5t^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{X}_{cl}(0) = \mathbf{I}_4. \qquad (3\text{-}27)$$

Using (3-26), the state transition matrix $\mathbf{\Phi}_o(t,0)$ is obtained as

$$\mathbf{\Phi}_o(t,0) = \mathbf{X}_o(t)\mathbf{X}_o^{-1}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0.5t^4 & 1.5t^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{\Phi}_o(t,0) = \mathbf{I}_4 . \tag{3-28}$$

Equations (3-27) and (3-14) satisfy, respectively, the matrix differential equations:

$$\dot{\mathbf{X}}_o(t) = \mathbf{A}_o(t)\mathbf{X}_o(t), \quad \mathbf{X}_o(0), \tag{3-29}$$

$$\dot{\mathbf{\Phi}}_o(t,0) = \mathbf{A}_o(t)\mathbf{\Phi}_o(t,0), \quad \mathbf{\Phi}_o(0,0) = \mathbf{I}_4, \tag{3-30}$$

which confirm that $\mathbf{X}_0(t)$ and $\mathbf{\Phi}_o(t,0)$ solve (3-29) and (3-30), respectively.

4 Conclusions

Determination of the analytical solutions for the fundamental and state transition matrices associated with linear time-varying (LTV) systems of the form given by (5), which is a matrix exponential, was investigated. It is well known that determining a matrix exponential is a difficult task. The investigation consisted of two types of LTV systems, namely, feedback control systems and observers. For the design of LTV control systems, one of the main objectives was to require the closed-loop system matrices to have specific structures, specifically, being upper or lower triangular matrices with identical elements on their main diagonal. The same objective was imposed on the observer matrices. Upper and lower triangular matrices have many desirable properties, such as they are commutative as wasstated in fact F.2 in the paper. The commutativity of a matrix will ease the determination of its fundamental and state transition matrices. Examples were given to demonstrate the analysis and design. Simulations and Matlab solutions using its command **EXPM** were provided as well. Future research will address disturbance cancellation control of LTV systems and the design of unknown input LTV observers.

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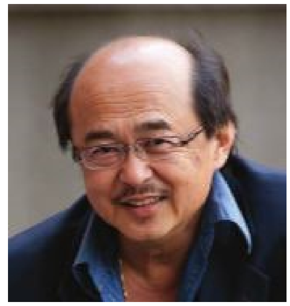

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